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THE EULER LINE IN NON-EUCLIDEAN GEOMETRY

A Thesis
Presented to the
Faculty of
California State University,
San Bernardino

In Partial Fulfillment
of the Requirements for the Degree
Master of Arts
in
Mathematics

by
Elena Strzheletska
December 2003

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A Thesis

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ABSTRACT

In Euclidean geometry, the circumcenter and the centroid of a nonequilateral triangle determine a line called the Euler line. The orthocenter of the triangle, the point of intersection of the altitudes, also belongs to this line. The main purpose of this thesis is to explore the conditions of the existence and the properties of the Euler line of a triangle in the hyperbolic plane. As a model of hyperbolic geometry Poincaré's conformal disk model is going to be used. For the algebraic representation of the geometric objects on the hyperbolic plane, we chose to use Hermitian matrices. In this paper we will find Hermitian representations of the three important points of a hyperbolic triangle (the circumcenter, centroid and orthocenter) as well as the Hermitian matrix that represents the Euler line, and discuss the conditions of their existence. We will complete our discussion by proving that if the circumcenter and orthocenter of the hyperbolic triangle exist, the Euler line goes through the orthocenter if and only if the triangle is isosceles.

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CHAPTER ONE

HISTORY AND BACKGROUND

1.1 History of Non-Euclidean Geometry

The notion that there exists a geometry different from Euclidean geometry is relatively new for mathematics. It was not until the 19th century that "a new geometry contrary to Euclid's" was found to be "logically possible." [12] Consider the Euclidean Parallel Postulate:

"If a straight line falling on two straight lines makes the two interior angles on the same side of it taken together less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles together less than two right angles." [9]

This proposition was stated in Euclid's *Elements* as the fifth (parallel) postulate. Modern books on geometry more often use the equivalent of the postulate: Through a point not on a given line there is a unique straight line parallel to the given line. The discussion about the parallel postulate lasted over 2000 years. Thinking that the fifth postulate depended on the other four, scientists continually tried to prove it. Non-Euclidean geometry was born as soon as the question about the independence of Euclid's parallel postulate was resolved.

One of the world's greatest mathematicians, Karl Friedrich Gauss (1777-1855), was probably the first (around 1813) to discover the existence of consistent geometry where Euclid's postulate was replaced by the contrary statement. To such a geometry, Gauss gave the name "non-Euclidean". Gauss would eventually be considered the founder

of non-Euclidean geometry, even though the German mathematician, physicist and astronomer, with international reputation, was fearful of being misunderstood, and never published his discovery. Later on, Ferdinand Schweikart (1780-1859), professor of jurisprudence, basically came to the same conclusion as Gauss, but also failed to publish his views.

The great Russian mathematician Nicolay Lobachevskiy (1792-1896) approached the problem in a different way. To show the dependence of the parallel postulate on the other four axioms, he chose to use the proof by contradiction. His thought was the assumption that through a given point not on a given line could go at least two straight lines parallel to the given line will lead him to a contradiction. The longer he worked, the more he believed that there was no contradiction in the new geometry he discovered. Having realized that geometry built on the assumption that Euclid's parallel axiom was not true does not contain a contradiction, Lobachevsky began to develop a new non-Euclidean geometry, and made a great impact on its foundation. In 1830 Lobachevsky was the first to publish a work paper on non-Euclidean geometry, On the Principles of Geometry. Even though the Hungarian mathematician János Bolyai (1802-1860) had actually made the same discovery of the new geometry before Lobachevsky, his was published later, in 1832, and only as an appendix to his father, Farcas Bolyai's, mathematical book.

That is why, during a 20-year period, non-Euclidean geometry was independently discovered four times. Thus, history gives credit for the discovery to all four mathematicians, and this geometry is sometimes called "Gauss-Schweikart-Bolyai-Lobachevskian," but is most often called "Bolyai-Lobachevskian," giving special recognition to those scientists who were willing to publish their findings. As with

everything new, non-Euclidean geometry was not understood and accepted for some time, even by advanced mathematicians, and Lobachevsky's work was even called "a satire" and "a caricature" of geometry.

The next phase in the evolution of non-Euclidean geometry began with developing its models. During the second half of the 19th century a number of models of hyperbolic geometry were introduced by Eugenio Beltrami, Felix Klein and Henri Poincaré. (In 1871, German mathematician Felix Klein proposed to call Euclidean, Bolyai-Lobachevsky and Riemann geometry "parabolic", "hyperbolic" and "elliptic" respectively). For this paper, we shall focus on one of the two developed by French mathematician and physicist Henri Poincaré (1854-1912) – his conformal disk model.

1.2. Poincaré's Version of Hyperbolic Geometry – Conformal Disk Model

To obtain this model, we will use a stereographic projection. Let us consider a unit hyperboloid of two sheets H^2 , and a unit circle \mathcal{C} that belongs to the generalized complex plane $\hat{\mathbb{C}}$. Let also S be the South Pole of H^2 . Under the stereographic projection of H^2 , from the point S to the generalized complex plane $\hat{\mathbb{C}}$, the points of the upper and lower planes of H^2 become the points of the interior and exterior of the unit circle respectively. The pairs of antipodal points of the upper and lower sheets of H^2 become the pairs of points on $\hat{\mathbb{C}}$ that map to each other under the inversion in \mathcal{C} . A section of a plane that goes through S and has common points with the hyperboloid is a hyperbola. Under the projection the image of the hyperbola is either an arc of a circle, orthogonal to the unit circle (if the plane does not contain z -axis), or a straight line that goes through the origin (otherwise). The last point we should make is that the stereographic projection preserves

the angles (it is conformal), but changes the distances. Now that we have described how to obtain Poincaré's conformal disk model, let us look at the model itself.

The points or d -points in Poincaré's model of hyperbolic geometry consist of those points in the unit disk \mathfrak{D} that do not belong to its boundary. "A d -line is that part of a (Euclidean) generalized circle, which meets \mathfrak{C} at right angles and which lies in \mathfrak{D} ". Two d -lines are called parallel if the generalized circles that afford them have a common point on the boundary of \mathfrak{D} , and ultra-parallel if the generalized circles that afford them do not meet either in \mathfrak{D} or on \mathfrak{C} . "A d -triangle consists of three points in the unit disc \mathfrak{D} that do not lie on a single d -line, together with the segments of the three d -lines joining them". [1]

We also want to state a few basic definitions, theorems and formulas. To find the non-Euclidean distance $d(z_1, z_2)$ between the points z_1 and z_2 in the unit disk \mathfrak{D} , we use the formula:

$$(1.1) \quad d(z_1, z_2) = th^{-1} \left(\frac{z_2 - z_1}{1 - \bar{z}_1 z_2} \right).$$

(For ease of the notations, shx , chx , thx , $cthx$, $schx$ will stand for hyperbolic sine, hyperbolic cosine, hyperbolic tangent, hyperbolic cotangent and hyperbolic secant respectively. It follows from (1.1) that the non-Euclidean distance $d(0, z)$ between the points 0 and z in the unit disk \mathfrak{D} is

$$(1.2) \quad d(0, z) = th^{-1}(|z|).$$

Let z_1 and z_2 be two points in the unit disk. There exists a unique d -line through z_1 and z_2 , and the center η of the circle affording this d -line could be found as follows:

$$(1.3) \quad \eta = \frac{(z_1 - z_2) - z_1 z_2 (\bar{z}_1 - \bar{z}_2)}{z_1 \bar{z}_1 - z_2 \bar{z}_2}.$$

Let p and q be two distinct points in the unit disk. Let ζ be the center of the circle that affords the perpendicular bisector of the d -line through p and q . Then

$$(1.4) \quad \zeta = \frac{(p - q) + pq(\bar{p} - \bar{q})}{p\bar{p} - q\bar{q}},$$

provided that $|p| \neq |q|$. In the case when $|p| = |q| = 0$, the perpendicular bisector to the d -line through p and q is a straight line - the diameter of the unit disk, that lies on the bisector of the angle pOq .

1.3 Hermitian Matrices

A complex 2×2 matrix H is Hermitian, provided that $H = H^*$, where H^* is the conjugate transpose of H . It follows that

$$(1.5) \quad H = \begin{pmatrix} A & B \\ \bar{B} & D \end{pmatrix},$$

where $A, D \in \mathbb{R}$, and $B \in \hat{\mathbb{C}}$.

Proposition.

The following form allows us to represent a circle in $\hat{\mathbb{C}}$ as the locus of all $z \in \hat{\mathbb{C}}$ such that

$$(1.6) \quad H(z) = \begin{pmatrix} z & 1 \end{pmatrix} \begin{pmatrix} A & B \\ \bar{B} & D \end{pmatrix} \begin{pmatrix} \bar{z} \\ 1 \end{pmatrix} = 0,$$

from which it follows that the locus is a circle in $\hat{\mathbb{C}}$ provided that $\det H < 0$.

From the above, we can make the following observations that could be easily verified:

1) Hermitian matrix

$$(1.7) \quad H_U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

represents the unit circle.

2) If $A = 0$, (1.5) represents a straight line.

3) Let ω and $k < \infty$ be respectively the center and the radius of some circle in $\hat{\mathbb{C}}$.

Let also

$$H = \begin{pmatrix} A & B \\ \overline{B} & D \end{pmatrix}$$

be a Hermitian matrix that represents this circle. Then

$$(1.8) \quad \begin{pmatrix} A & B \\ \overline{B} & D \end{pmatrix} = A \begin{pmatrix} 1 & -\overline{\omega} \\ -\omega & \omega\overline{\omega} - k^2 \end{pmatrix}.$$

4) According to the Proposition stated above, two Hermitian matrices H_1 and H_2 represent the same circle if and only if $H_1 = \lambda H_2$, where λ is some nonzero real number. In our computations we will be interested in Hermitian matrices as representatives of circles, thus, though strictly speaking, two matrices H_1 and H_2 such that $H_1 = \lambda H_2$ are not equal, we won't make a distinction between them. To stress this real - projective equivalence of Hermitian matrices, instead of usual parentheses

$$\begin{pmatrix} A & B \\ \overline{B} & D \end{pmatrix}$$

we will use square brackets:

$$\begin{bmatrix} A & B \\ \overline{B} & D \end{bmatrix}.$$

Let \check{H}_2 denote the Hermitian matrix adjoint to H . The inner product of two Hermitian matrices H_1 and H_2 is defined by

$$(1.9) \quad \langle H_1, H_2 \rangle = \text{Tr}(H_1 \check{H}_2),$$

where

$$(1.10) \quad \text{Tr}(H_1 \check{H}_2) = A_1 D_2 + A_2 D_1 - B_1 \overline{B}_2 - B_2 \overline{B}_1.$$

It follows that if Hermitian matrices H_1 and H_2 represent circles orthogonal to the unit circle:

$$1) \langle H_1, H_2 \rangle = 2A_1 A_2 - B_1 \overline{B}_2 - B_2 \overline{B}_1$$

$$2) \langle H_1, H_1 \rangle = 2(A_1^2 - B_1^2) = 2 \det H_1$$

Since any circle that affords a d-line is orthogonal to the unit circle, it is represented by Hermitian matrix

$$(1.11) \quad H = \begin{bmatrix} A & B \\ \overline{B} & A \end{bmatrix}.$$

Let C_1 and C_2 be two generalized circles, that are represented by Hermitian matrices H_1 and H_2 , respectively. The angle φ between C_1 and C_2 we define as follows:

$$(1.12) \quad \cos \varphi = \frac{\langle H_1, H_2 \rangle}{\sqrt{\langle H_1, H_1 \rangle \cdot \langle H_2, H_2 \rangle}}.$$

We can conjecture from (1.12) that:

- 1) C_1 and C_2 intersect if and only if $|\cos \varphi| < 1$,
in particular C_1 and C_2 are orthogonal if and only if $\cos \varphi = 0$,
- 2) C_1 and C_2 are tangent to each other if and only if $|\cos \varphi| = 1$,
- 3) C_1 and C_2 do not have common points if and only if $|\cos \varphi| > 1$.

1.4 Pencils of Circles

Euclidean geometry considers two kinds of pencils – concurrent and parallel. In hyperbolic geometry there are three kinds of pencils. In addition to elliptic and parabolic pencils, which are analogous to the concurrent and parallel Euclidean pencils, respectively, there exist hyperbolic pencils of circles. Any two distinct circles on the hyperbolic plane generate the pencil of circles. If C_1 and C_2 are two circles in \hat{C} , then any circle C that belongs to the pencil defined by C_1 and C_2 can be expressed as a linear combination of the two generators of the pencil. In other words,

$$(1.13) \quad C = \lambda C_1 + \mu C_2,$$

where $\lambda, \mu \in \mathbb{R}$.

1.5 Notations

- 1) We will use the small letterers: h, k , etc. to denote Euclidean measures and capital letterers: O, H, K to state the vertices or d -points and non-Euclidean measures - the context will determined which usage is intended.
- 2) Sometimes, when it is understood from the context, we will call the circles by the names of d -lines they afford.

3) We will use

$$(1.14) \quad \rho^* = \frac{1}{\rho}$$

for the inversion of ρ in the unit circle.

CHAPTER TWO

A HYPERBOLIC TRIANGLE AND ITS CEVIANS

2.1 A Hyperbolic Triangle

Let HOK be a hyperbolic triangle in the unit disk. Without loss of generality, we can assume that one of the vertices of the triangle, say O coincides with the origin, and the side HO is on the real axis. Let also the sides HO and KO have the Euclidean lengths h and k respectively ($h > k$), and the angle KOH equal θ . Then the vertices of the triangle have coordinates: $O = 0$, $H = h$, $K = ke^{i\theta}$. The two sides of the triangle KO and HO are straight lines. They are represented by the Hermitian matrices

$$H_{HO} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix},$$

$$H_{KO} = \begin{bmatrix} 0 & \sin\theta + i\cos\theta \\ \sin\theta - i\cos\theta & 0 \end{bmatrix}.$$

A d-line HK is an arc of a (Euclidean) circle. To find the center η_{HK} of this circle, we use

(1.3), where $z_1 = h$ and $z_2 = ke^{i\theta}$:

$$\begin{aligned} \eta_{HK} &= \frac{(h - ke^{i\theta}) - hke^{i\theta}(h - ke^{-i\theta})}{hk(e^{-i\theta} - e^{i\theta})} \\ &= \frac{h(1 + k^2) - ke^{i\theta}(1 + h^2)}{hk(e^{-i\theta} - e^{i\theta})} \end{aligned}$$

Given that $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$,

$$\eta_{HK} = \frac{h(1 + k^2) - k(\cos\theta + i\sin\theta)(1 + h^2)}{hk(\cos\theta - i\sin\theta - \cos\theta - i\sin\theta)},$$

$$\eta_{HK} = \frac{k(1+h^2)\sin\theta + i(h(1+k^2) - k(1+h^2)\cos\theta)}{hk(\cos\theta - i\sin\theta - \cos\theta - i\sin\theta)}$$

Let

$$(2.1) \quad k(1+h^2) = k^+ \quad \text{and} \quad h(1+k^2) = h^+.$$

Then from the last expression

$$(2.2) \quad \eta_{HK} = \frac{k^+ \sin\theta + i(h^+ - k^+ \cos\theta)}{2hk \sin\theta}.$$

According to (1.8) and (2.2), the expression for the Hermitian matrix of a d -line that is obtained from the point η_{HK} can be written in a form:

$$(2.3) \quad H_{HK} = \begin{bmatrix} 1 & -\overline{\eta}_{HK} \\ -\eta_{HK} & 1 \end{bmatrix},$$

$$(2.3) \quad H_{HK} = \begin{bmatrix} 1 & \frac{-k^+ \sin\theta + i(h^+ - k^+ \cos\theta)}{2hk \sin\theta} \\ \frac{-k^+ \sin\theta - i(h^+ - k^+ \cos\theta)}{2hk \sin\theta} & 1 \end{bmatrix}$$

Multiplying the last matrix by $2hk \sin\theta \in \mathbb{R}$, we obtain another matrix that by (1.6) also represents H_{HK} . And so:

$$(2.4) \quad H_{HK} = \begin{bmatrix} 2hk \sin\theta & -k^+ \sin\theta + i(h^+ - k^+ \cos\theta) \\ -k^+ \sin\theta - i(h^+ - k^+ \cos\theta) & 2hk \sin\theta \end{bmatrix}.$$

In our further calculations we may use Hermitian representations of H_{HK} of either form (2.3) or (2.4), depending on which is more convenient.

Now that we have found the Hermitian representations for the sides of the triangle HKO , we can calculate its angles H and K , but first we need to compute the inner

products $\langle H_1, H_1 \rangle$, $\langle H_2, H_2 \rangle$, and $\langle H_1, H_2 \rangle$. By (1.9) and (1.10)

$$\langle H_1, H_1 \rangle = -2,$$

$$\langle H_2, H_2 \rangle = -2((h^+)^2 + (k^+)^2 - 2h^+k^+ \cos \theta - 4h^2k^2 \sin^2 \theta),$$

$$\langle H_1, H_2 \rangle = -2(h^+ - k^+ \cos \theta).$$

Hence by (1.12)

$$(2.5) \quad \cos H = \frac{(h^+ - k^+ \cos \theta)}{\sqrt{(h^+)^2 + (k^+)^2 - 2h^+k^+ \cos \theta - 4h^2k^2 \sin^2 \theta}}$$

and

$$(2.6) \quad \cos K = \frac{(k^+ - h^+ \cos \theta)}{\sqrt{(h^+)^2 + (k^+)^2 - 2h^+k^+ \cos \theta - 4h^2k^2 \sin^2 \theta}}.$$

2.2 Perpendicular Bisectors

Let P_{HK} , P_{HO} and P_{KO} be perpendicular bisectors to the sides HK , HO and KO , respectively. Let also the centers of the circles that afford the perpendicular bisectors be ζ_{HK} , ζ_{HO} , ζ_{KO} , which we can calculate using (1.4):

$$\begin{aligned} \zeta_{HK} &= \frac{(ke^{i\theta} - h) + khe^{i\theta}(ke^{-i\theta} - h)}{k^2e^{i\theta}e^{-i\theta} - h^2} \\ &= \frac{ke^{i\theta} - h + k^2h - kh^2e^{i\theta}}{k^2 - h^2} \\ &= \frac{ke^{i\theta}(1 - h^2) - h(1 - k^2)}{k^2 - h^2}, \end{aligned}$$

and finally

$$\zeta_{HK} = \frac{k^- e^{i\theta} - h^-}{k^2 - h^2}.$$

$$\zeta_{HO} = \frac{1}{h},$$

$$\zeta_{KO} = \frac{1}{ke^{-i\theta}} = \frac{e^{i\theta}}{k}.$$

In the formula for ζ_{HK} we used substitutions:

$$(2.7) \quad h(1 - k^2) = h^- \text{ and } k(1 - h^2) = k^-.$$

Using (1.11) and the properties of Hermitian matrices, we obtain the expressions for P_{HK} ,

P_{HO} and P_{KO} :

$$H_{P_{HK}} = \begin{bmatrix} k^2 - h^2 & h^- - k^- e^{-i\theta} \\ h^- - k^- e^{i\theta} & k^2 - h^2 \end{bmatrix},$$

$$H_{P_{HO}} = \begin{bmatrix} h & -1 \\ -1 & h \end{bmatrix},$$

$$H_{P_{KO}} = \begin{bmatrix} k & -e^{-i\theta} \\ -e^{i\theta} & k \end{bmatrix}.$$

Now we shall discuss the question of whether all three perpendicular bisectors belong to the same pencil of circles. According to (1.13), in order to prove that $H_{P_{HO}}$, $H_{P_{KO}}$ and $H_{P_{HK}}$ are the members of the same pencil we need to find λ and $\mu \in \mathbb{R}$ such that

$$(2.8) \quad \lambda H_{P_{HO}} + \mu H_{P_{KO}} = H_{P_{HK}}.$$

The left hand side of (2.8) :

$$\begin{aligned}\lambda H_{P_{HO}} + \mu H_{P_{KO}} &= \lambda \begin{bmatrix} h & -1 \\ -1 & h \end{bmatrix} + \mu \begin{bmatrix} k & -e^{-i\theta} \\ -e^{i\theta} & k \end{bmatrix} \\ &= \begin{bmatrix} \lambda h + \mu k & -\lambda - \mu e^{-i\theta} \\ -\lambda - \mu e^{i\theta} & \lambda h + \mu k \end{bmatrix}\end{aligned}$$

Comparing the last matrix with the matrix for P_{HK} , we can easily find the values for λ and μ such that (2.10) holds. When $\lambda = -h^-$, and $\mu = k^-$, $H_{P_{HK}}$ is a linear combination of $H_{P_{HO}}$ and $H_{P_{KO}}$. Hence we have proved the following theorem.

Theorem 2.1.

The perpendicular bisectors of the sides of a hyperbolic triangle belong to the same pencil of circles.

Corollary

The circumcenter of a hyperbolic triangle exists if and only if the pencil of the perpendicular bisectors is elliptic. Figure (2.1) shows $\triangle HKO$, whose perpendicular bisectors are concurrent in the circumcenter of the triangle.

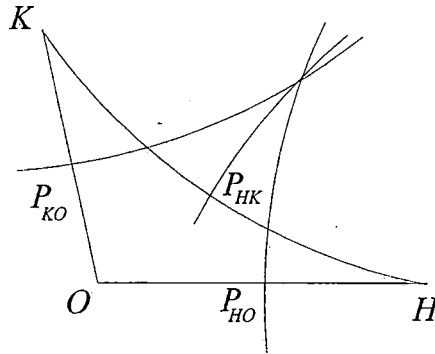


Figure 2.1

2.3 Medians

Let M_H , M_K and M_O be the non-Euclidean midpoints of KO , HO and HK , respectively. Since the Euclidean distance between the points H and O is h and the non-Euclidean distance is H it follows from (1.2), that

$$(2.9a) \quad h = thH,$$

and similarly

$$(2.9b) \quad k = thK,$$

We need to express the distance m_K between the origin and the non-Euclidean midpoint M_H in Euclidean terms. The non-Euclidean distance between the O and M_H is $\frac{1}{2}H$,

hence

$$\begin{aligned} m_{HO} &= th\left(\frac{H}{2}\right) = \frac{chH-1}{shH} \\ &= \frac{1 - \sqrt{1 - th^2H}}{thH} \\ &= \frac{1 - \sqrt{1 - h^2}}{h}. \end{aligned}$$

Thus

$$m_{HO} = \frac{1}{h}(1 - \sqrt{1 - h^2}).$$

Using similar arguments and considering the fact that the angle between KO and the real axis is θ , we find

$$m_{KO} = \frac{1}{k}(1 - \sqrt{1 - k^2})e^{i\theta}.$$

It would be much more complicated to calculate the third midpoint m_{HK} , but fortunately there is no need to do it, and the reason for it we will explain later in this chapter. Knowing the expressions for m_{HO} and m_{KO} , we can write the equations of two medians HM_H and KM_K , and then find OM_O as a linear combination of the first two.

We will start from finding η_{KM_K} , the center of the circle that affords d -line KM_K .

To do that, we use the formula (1.3), where $z_1 = K = ke^{i\theta}$, and

$$z_2 = M_K = \frac{1}{h}(1 - \sqrt{1 - h^2}). \text{ Then}$$

$$\begin{aligned} \eta_{KM_K} &= \frac{ke^{i\theta} - \frac{1 - \sqrt{1 - h^2}}{h} - \frac{ke^{i\theta}(1 - \sqrt{1 - h^2})}{h}(ke^{-i\theta} - \frac{1 - \sqrt{1 - h^2}}{h})}{\frac{k(e^{i\theta} - e^{-i\theta})(1 - \sqrt{1 - h^2})}{h}} \\ &= \frac{h^2 ke^{i\theta} - h(1 - \sqrt{1 - h^2}) - (1 - \sqrt{1 - h^2})ke^{i\theta}(hke^{-i\theta} - (1 - \sqrt{1 - h^2}))}{h(1 - \sqrt{1 - h^2})k(e^{i\theta} - e^{-i\theta})} \\ &= \frac{ke^{i\theta}(h^2 + (1 - \sqrt{1 - h^2})^2) - h(1 - \sqrt{1 - h^2})(1 + k^2)}{h(1 - \sqrt{1 - h^2})k(e^{i\theta} - e^{-i\theta})} \\ &= \frac{2ke^{i\theta}(1 - \sqrt{1 - h^2}) - h^+(1 - \sqrt{1 - h^2})}{h(1 - \sqrt{1 - h^2})k(e^{i\theta} - e^{-i\theta})} \\ &= \frac{2ke^{i\theta} - h^+}{hk(e^{i\theta} - e^{-i\theta})}. \end{aligned}$$

Multiplying by i the numerator and the denominator of the last expression, and recalling that $e^{i\theta} = \cos\theta + i\sin\theta$, we obtain:

$$\eta_{KM_K} = \frac{2k\sin\theta + i(2k\cos\theta - h^+)}{2hk\sin\theta}.$$

Then

$$H_{KM_K} = \begin{bmatrix} 2hk \sin \theta & -2k \sin \theta + i(h^+ - 2k \cos \theta) \\ -2k \sin \theta - i(h^+ - 2k \cos \theta) & 2hk \sin \theta \end{bmatrix}.$$

Performing similar calculations, we find

$$\eta_{HM_H} = \frac{k^+ \sin \theta + i(2h - k^+ \cos \theta)}{2hk \sin \theta},$$

which leads to

$$H_{HM_H} = \begin{bmatrix} 2hk \sin \theta & -k^+ \sin \theta + i(2h - k^+ \cos \theta) \\ -k^+ \sin \theta - i(2h - k^+ \cos \theta) & 2hk \sin \theta \end{bmatrix}$$

Now we will return to the question of the third median. Let us first state a theorem.

Theorem 2.2

The medians of a hyperbolic triangle belong to the same pencil of circles. The pencil is elliptic.

Before we start proving this theorem, some preliminary work should be done.

Definition 2.1

If A , B and X are distinct points on a d -line, then their hyperbolic ratio is

$$h(A, X, B) = \frac{sh(d(A, X))}{sh(d(X, B))}, \text{ if } X \text{ is between } A \text{ and } B, \text{ and}$$

$$h(A, X, B) = -\frac{sh(d(A, X))}{sh(d(X, B))}, \text{ otherwise [13].}$$

Lemma 2.1 (Converse of Ceva's Theorem)

Let ABC be a hyperbolic triangle, and P, Q, R three points in the unit disk. If P lies on the d -line AB , Q on BC and R on CA in such a way that

$$h(A, P, B) \cdot h(B, Q, C) \cdot h(C, R, A) = 1,$$

and two of the d -lines CP , BR and AQ meet, then all three are concurrent [13].

Lemma 2.2

Let l_1 and l_2 be two circles, orthogonal to the unit circle. Then any generalized circle of the pencil generated by l_1 and l_2 is also orthogonal to the unit circle.

Proof. Let H_1, H_2 be the two Hermitian matrices that represent l_1 and l_2 . Since l_1 and l_2 are orthogonal to \mathfrak{C} , H_1 and H_2 have the following forms:

$$H_1 = \begin{bmatrix} A_1 & B_1 \\ \overline{B_1} & A_1 \end{bmatrix} \quad \text{and} \quad H_2 = \begin{bmatrix} A_2 & B_2 \\ \overline{B_2} & A_2 \end{bmatrix}$$

Any circle from the pencil that is generated by H_1, H_2 is of the form

$$\lambda H_1 + \mu H_2,$$

where $\lambda, \mu \in \mathbb{R}$.

$$\begin{aligned} \lambda H_1 + \mu H_2 &= \lambda \begin{bmatrix} A_1 & B_1 \\ \overline{B_1} & A_1 \end{bmatrix} + \mu \begin{bmatrix} A_2 & B_2 \\ \overline{B_2} & A_2 \end{bmatrix} \\ &= \begin{bmatrix} \lambda A_1 + \mu A_2 & \lambda B_1 + \mu B_2 \\ \overline{\lambda B_1 + \mu B_2} & \lambda A_1 + \mu A_2 \end{bmatrix}, \end{aligned}$$

and the Lemma is proved since the last matrix represents the circle orthogonal to the unit circle. Now we can prove Theorem 2.2.

Proof . Let us use triangle HKO as an example of a hyperbolic triangle. Consider the pencil of circles generated by the two circles affording HM_H and KM_K . HM_H and KM_K have a common point (say point M) inside of the unit disk, which leads us to two

conclusions. First, by the Lemma 2.1, all three medians are concurrent. Second, the pencil is elliptic, and the only straight line of the pencil goes through M . By Lemma 2.2, the straight line is a d-line, and hence goes through the origin. That means that OM_O , the third median of the triangle, is coincide with OM , the straight line of the pencil, and thus belongs to it, which proves the Theorem.

Corollary

The three medians of a hyperbolic triangle are concurrent. You can see the illustration to the Corollary in Figure (2.2).

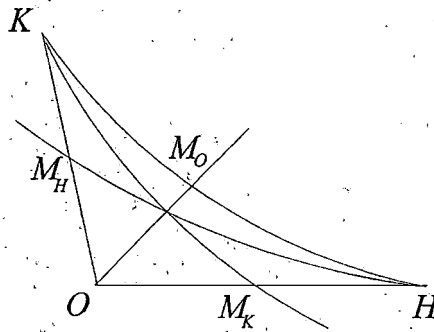


Figure 2.2

It follows from Theorem 2.2 that $\lambda H_{KM_K} + \mu H_{HM_H} = H_{OM_O}$ for some λ and μ .

Since OM_O is a straight line that goes through the origin, it is represented by the matrix H_{OM_O} of the form:

$$H_{OM_O} = \begin{bmatrix} 0 & B \\ \overline{B} & 0 \end{bmatrix}.$$

Now, from the equation

$$\begin{aligned}
& \lambda \begin{bmatrix} 2hk \sin \theta & -2k \sin \theta + i(h^+ - 2k \cos \theta) \\ -2k \sin \theta - i(h^+ - 2k \cos \theta) & 2hk \sin \theta \end{bmatrix} + \\
& + \mu \begin{bmatrix} 2hk \sin \theta & -k^+ \sin \theta + i(2h - k^+ \cos \theta) \\ -k^+ \sin \theta - i(2h - k^+ \cos \theta) & 2hk \sin \theta \end{bmatrix} \\
& = \begin{bmatrix} 0 & B \\ \bar{B} & 0 \end{bmatrix}
\end{aligned}$$

the following conclusions can be made:

$$1) (\lambda + \mu)2hk \sin \theta = 0 \quad \text{and}$$

$$2) \lambda(-2k \sin \theta - i(h^+ - 2k \cos \theta)) + \mu(-k^+ \sin \theta + i(2h - k^+ \cos \theta)) = B.$$

The first equation immediately yields $\mu = -\lambda$, and substituting this into the second we obtain

$$\lambda(-2k \sin \theta - i(h^+ - 2k \cos \theta)) - \lambda(-k^+ \sin \theta + i(2h - k^+ \cos \theta)) = B.$$

Then

$$\begin{aligned}
B &= \lambda(-2k \sin \theta + i(h^+ - 2k \cos \theta) + k^+ \sin \theta - i(2h - k^+ \cos \theta)) \\
&= -2k \sin \theta + k^+ \sin \theta + i(h^+ - 2k \cos \theta - 2h + k^+ \cos \theta) \\
&= -k \sin \theta(1 - h^2) + i(-k \cos \theta(1 - h^2) - h(1 - k^2)) = -k^- \sin \theta + i(k^- \cos \theta + h^-)
\end{aligned}$$

Therefore the third median OM_O is represented by the matrix

$$H_{OM_O} = \begin{bmatrix} 0 & -k^- \sin \theta + i(h^- + k^- \cos \theta) \\ -k^- \sin \theta - i(h^- + k^- \cos \theta) & 0 \end{bmatrix}.$$

2.4 Altitudes

Let A_H, A_K and A_O be the feet of the altitudes of triangle HKO from the vertices H, K and O , respectively. Our goal is to find the Hermitian matrices that represent the altitudes. We are going to start from OA_O , which is a straight line and hence its matrix H_{OA_O} is of the form

$$H_{OA_O} = \begin{bmatrix} 0 & B \\ \overline{B} & 0 \end{bmatrix},$$

where $B \in \widehat{\mathbb{C}}$. Let $B = b_1 + ib_2 = \text{Re } B + i \text{Im } B$.

Since OA_O is an altitude of HK , OA_O goes through the center η_{HK} of the circle HK , and thus B has to satisfy the following equation:

$$\begin{pmatrix} \eta_{HK} & 1 \end{pmatrix} \begin{bmatrix} 0 & B \\ \overline{B} & 0 \end{bmatrix} \begin{pmatrix} \overline{\eta}_{HK} \\ 1 \end{pmatrix} = 0.$$

Then multiplying through the matrices of the last expression and substituting B for $b_1 + ib_2$, we obtain the equivalent equation

$$(b_1 - ib_2)\overline{\eta}_{HK} + (b_1 + ib_2)\eta_{HK} = 0,$$

which simplifies to

$$b_1 \text{Re}(\eta_{HK}) - b_2 \text{Im}(\eta_{HK}) = 0.$$

If we now recall that

$$\text{Re}(\eta_{HK}) = \frac{k^+ \sin \theta}{2hk \sin \theta} \quad \text{and} \quad \text{Im}(\eta_{HK}) = \frac{h^+ - k^+ \cos \theta}{2hk \sin \theta},$$

then we receive the equation

$$b_1 \frac{k^+ \sin \theta}{2hk \sin \theta} - b_2 \frac{h^+ - k^+ \cos \theta}{2hk \sin \theta} = 0.$$

It follows then that

$$b_1 = \frac{h^+ - k^+ \cos \theta}{k^+ \sin \theta} b_2.$$

We shall use the fact that $OA_O \perp HK$ one more time. As it follows from (1.9), the inner product of two orthogonal circles is zero, so

$$\langle H_{HK}, H_{OA_O} \rangle = 0.$$

The last condition is equivalent to the following equation:

$$\begin{aligned} & (k^+ \sin \theta - i(h^+ - k^+ \cos \theta)) \left(\frac{(1 + h^2)}{2h} + ib_2 \right) - \\ & - \left(\frac{(1 + h^2)}{2h} - ib_2 \right) (k^+ \sin \theta + i(h^+ - k^+ \cos \theta)) = 0, \end{aligned}$$

which gives us the solution for b_2 :

$$b_2 = \frac{-(1 + h^2)k^+ \sin \theta}{2h(h^+ - k^+ \cos \theta)}.$$

Then

$$b_1 = \frac{-(1 + h^2)}{2h},$$

and finally,

$$B = \frac{-(1 + h^2)}{2h(h^+ - k^+ \cos \theta)} ((h^+ - k^+ \cos \theta) + ik^+ \sin \theta).$$

Using the properties of Hermitian matrices, we write H_{OA_O} as follows:

$$H_{OA_O} = \begin{bmatrix} 0 & (h^+ - k^+ \cos \theta) + ik^+ \sin \theta \\ (h^+ - k^+ \cos \theta) - ik^+ \sin \theta & 0 \end{bmatrix}.$$

Now, applying the condition for HA_H and KO to be orthogonal, and for KA_K and HO to be orthogonal, and performing the calculations very similar to the case with OA_O , we obtain:

$$H_{HA_H} = \begin{bmatrix} -2h \cos \theta & ((h^2 + 1) \cos \theta - i(h^2 + 1) \sin \theta) \\ ((h^2 + 1) \cos \theta + i(h^2 + 1) \sin \theta) & -2h \cos \theta \end{bmatrix}$$

and

$$H_{KA_K} = \begin{bmatrix} -2k \cos \theta & k^2 + 1 \\ k^2 + 1 & -2k \cos \theta \end{bmatrix}.$$

As in the case with the perpendicular bisectors, we want to know if all the three altitudes belong to the same pencil or in other words is there such λ and $\mu \in \mathbb{R}$, that

$$(2.10) \quad \lambda H_{HA_H} + \mu H_{KA_K} = H_{OA_O} ?$$

Comparing the three expressions for the matrices of the altitudes, we observe that for (2.10) to be true, it is necessary that

$$1) \quad \lambda h + \mu k = 0$$

and

$$\begin{aligned} 2) \quad & \lambda((h^2 + 1) \cos \theta - i(h^2 + 1) \sin \theta) + \mu(k^2 + 1) \\ & = (h^+ - k^+ \cos \theta) + ik^+ \sin \theta. \end{aligned}$$

Solving the two equations simultaneously, we obtain the answer: if $\mu = h$ and $\lambda = -k$, then (2.10) holds. Thus we just proved another theorem.

Theorem 2.3.

The altitudes of a hyperbolic triangle belong to the same pencil of circles.

Corollary

The orthocenter of a hyperbolic triangle exists if and only if the pencil of the altitudes is elliptic. In figure (2.3) we have drawn $\triangle HKO$, whose altitudes are concurrent in the orthocenter of the triangle.

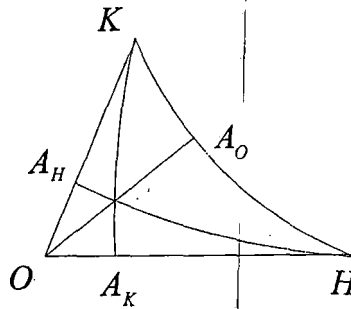


Figure 2.3

$$\frac{k\alpha_1}{h} - (\alpha_1 \cos \theta + \alpha_2 \sin \theta) = 0.$$

Now we express α_2 through α_1 to get:

$$\alpha_2 = \frac{\alpha_1(k - h \cos \theta)}{h \sin \theta},$$

and substitute this into (3.3):

$$\left(\alpha_1^2 + \alpha_1^2 \left(\frac{k - h \cos \theta}{h \sin \theta} \right)^2 + 1 \right) h - 2\alpha_1 = 0,$$

$$\frac{(\alpha_1^2(h^2 \sin^2 \theta + k^2 + h^2 \cos^2 \theta - 2hk \cos \theta) + h^2 \sin^2 \theta)h - 2\alpha_1 h^2 \sin^2 \theta}{h^2 \sin^2 \theta} = 0.$$

We notice that $h^2 \sin^2 \theta \neq 0$ and can replace the previous equation by its equivalent:

$$(3.6) \quad \alpha_1^2(h^2 + k^2 - 2hk \cos \theta) - 2\alpha_1 h \sin^2 \theta + h^2 \sin^2 \theta = 0.$$

Let us consider only the triangles for which $h^2 + k^2 - 2hk \cos \theta \neq 0$ (the case $h^2 + k^2 - 2hk \cos \theta = 0$ corresponds to the isosceles right triangle), and write (3.6) in non-Euclidean terms:

$$\alpha_1^2(th^2H + th^2K - 2thHthK \cos \theta) - 2\alpha_1 thH \sin^2 \theta + th^2H \sin^2 \theta = 0.$$

To simplify the notation, let

$$(3.7) \quad th^2H + th^2K - 2thHthK \cos \theta = f_\alpha(H, K),$$

and then we can rewrite the last equation as follows:

$$\alpha_1^2 f_\alpha(H, K) - 2\alpha_1 thH \sin^2 \theta + th^2H \sin^2 \theta = 0.$$

The last is a quadratic equation that we solve using quadratic formula:

$$\alpha_1 = \frac{thH \sin^2 \theta \pm thH |\sin \theta| \sqrt{\sin^2 \theta - f_\alpha(H, K)}}{f_\alpha(H, K)},$$

$$\alpha_1 = \frac{thH \sin \theta \left[\sin \theta \pm \sqrt{\sin^2 \theta - f_\alpha(H, K)} \right]}{f_\alpha(H, K)},$$

since $0 < \theta < \pi$. Considering that α_1 should belong to the unit disk ($|\alpha_1| < 1$), we have the only one solution for α_1 :

$$\alpha_1 = \frac{thH \sin \theta \left[\sin \theta - \sqrt{\sin^2 \theta - f_\alpha(H, K)} \right]}{f_\alpha(H, K)}.$$

Then

$$\alpha_2 = \frac{(thK - thH \cos \theta) \left[\sin \theta - \sqrt{\sin^2 \theta - f_\alpha(H, K)} \right]}{f_\alpha(H, K)},$$

and therefore

$$(3.8) \quad \alpha = \frac{\left[\sin \theta - \sqrt{\sin^2 \theta - f_\alpha(H, K)} \right]}{f_\alpha(H, K)} (thH \sin \theta + i(thK - thH \cos \theta)).$$

To simplify the notation of (3.8), let

$$\alpha_0 = thH \sin \theta + i(thK - thH \cos \theta).$$

Then

$$|\alpha_0|^2 = th^2 H + th^2 K - 2thHthK \cos \theta = f_\alpha(H, K)$$

and

$$\alpha = \frac{\left[\sin \theta - \sqrt{\sin^2 \theta - |\alpha_0|^2} \right]}{|\alpha_0|^2} \alpha_0.$$

Finally, applying (1.14), we obtain:

$$(3.9) \quad \alpha = \left[\sin \theta - \sqrt{\sin^2 \theta - |\alpha_0|^2} \right] \alpha_0^*$$

From the calculations above we excluded the case when $f_\alpha(H, K) = 0$, in which

$$\alpha = \frac{thH}{2}(1 + i).$$

Now that we know how to calculate the circumcenter of a triangle, let us state the following theorem.

Theorem 3.1.

If two perpendicular bisectors of the sides of a hyperbolic triangle are concurrent, then the point of their concurrence - the circumcenter of the triangle - is the center of the circumscribed circle.

The analytic proof of the theorem can be found in [13]. We could give an algebraic proof, which would consist of computing the distances between the circumcenter and the vertices of the triangle. After performing straightforward calculations, one can verify, that $d(\alpha, O) = d(\alpha, H) = d(\alpha, K) = R$, where R is the non-Euclidean circumradius. We will limit ourselves to finding the non-Euclidean distance $d(\alpha, O)$ between the points α and O . From (1.2) and (3.9)

$$d(\alpha, O) = |\alpha| = th^{-1} \left(\left[\sin \theta - \sqrt{\sin^2 \theta - |\alpha_0|^2} \right] |\alpha_0^*| \right).$$

Therefore, the radius R of the circle circumscribed about a hyperbolic triangle is

$$R = th^{-1} \left(\left[\sin \theta - \sqrt{\sin^2 \theta - |\alpha_0|^2} \right] |\alpha_0^*| \right).$$

3.2 Centroid

Let $\beta = \beta_1 + i\beta_2$ denote the centroid of a hyperbolic triangle. To find the centroid, we will use similar arguments that we used to obtain the formula for the circumcenter, that

is, find the centroid as a point of intersection of two medians HM_H and OM_O . Any point of the extended complex plane that belongs to the circle HM_H , has to satisfy the equation:

$$\begin{pmatrix} \beta & 1 \end{pmatrix} \begin{bmatrix} 2hk \sin \theta & -k^+ \sin \theta + i(2h - k^+ \cos \theta) \\ -k^+ \sin \theta - i(2h - k^+ \cos \theta) & 2hk \sin \theta \end{bmatrix} \begin{pmatrix} \bar{\beta} \\ 1 \end{pmatrix} = 0.$$

On the other hand, any point of \hat{C} that is on OM_O satisfies the equation:

$$\begin{pmatrix} \beta & 1 \end{pmatrix} \begin{bmatrix} 0 & -k^- \sin \theta + i(h^- + k^- \cos \theta) \\ -k^- \sin \theta - i(h^- + k^- \cos \theta) & 0 \end{bmatrix} \begin{pmatrix} \bar{\beta} \\ 1 \end{pmatrix} = 0$$

By simplifying these two equations we get:

$$(3.10) \quad hk \sin \theta (\beta_1^2 + \beta_2^2 + 1) - (k^+ \sin \theta \alpha_1 + (2h - k^+ \cos \theta) \beta_2) = 0$$

and

$$k^- \beta_1 \sin \theta - (h^- + k^- \cos \theta) \beta_2 = 0.$$

From the last equation,

$$\beta_2 = \frac{k^- \sin \theta}{h^- + k^- \cos \theta} \beta_1$$

After we simplify the equation (3.10) and substitute the expression for β_2 into it, we receive:

$$(3.11) \quad \left(\frac{\beta_1}{h^- + k^- \cos \theta} \right)^2 ((h^-)^2 + (k^-)^2 + 2k^- h^- \cos \theta) - \left(\frac{\beta_1}{h^- + k^- \cos \theta} \right) ((1 + h^2)(1 - k^2) + 2(1 - h^2)) + 1 = 0.$$

By employing the substitution

$$y = \frac{\beta_1}{h^- + k^- \cos \theta},$$

we simplify the equation (3.11):

$$y^2((h^-)^2 + (k^-)^2 + 2k^-h^- \cos \theta) - y((1 + h^2)(1 - k^2) + 2(1 - h^2)) + 1 = 0.$$

After this point it is going to be more convenient to use non-Euclidean distances instead of Euclidean. So, with the notation (2.9), we rewrite the last equation as:

$$\begin{aligned} y^2 \left(\frac{th^2 H}{ch^4 K} + \frac{th^2 K}{ch^4 H} + 2 \frac{thH}{ch^2 K} \frac{thK}{ch^2 H} \cos \theta \right) - y \left(\frac{1 + th^2 H}{ch^2 K} + \frac{2}{ch^2 H} \right) + 1 = 0, \\ y^2 \left(\frac{sh^2(2H) + sh^2(2K) + 2 sh(2H)sh(2K) \cos \theta}{4ch^4 H ch^4 K} \right) - \\ - y \left(\frac{ch(2H) + ch(2K) + 1}{ch^2 H ch^2 K} \right) + 1 = 0. \end{aligned}$$

Let us introduce the substitutions that will be used throughout the paper:

$$(3.12) \quad ch(2H) + ch(2K) + 1 = m$$

$$(3.13) \quad sh^2(2H) + sh^2(2K) + 2sh(2H)sh(2K) \cos \theta = f_B(2H, 2K)$$

With (3.12) and (3.13) we can rewrite our equation as follows:

$$y^2 \left(\frac{f_B(2H, 2K)}{4ch^4 H ch^4 K} \right) - y \left(\frac{m}{ch^2 H ch^2 K} \right) + 1 = 0,$$

which (since $chH \cdot chK \neq 0$) yields the quadratic equation

$$y^2 f_B(2H, 2K) - y(4m ch^2 H ch^2 K) + 4ch^4 H ch^4 K = 0.$$

This equation has two solutions:

$$y_{(1),(2)} = \frac{m \pm \sqrt{m^2 - f_B(2H, 2K)}}{f_B(2H, 2K)} \cdot 2ch^2 H ch^2 K.$$

Let us now express β in non-Euclidean terms:

$$\begin{aligned}\beta &= y \left(\left(\frac{thH}{ch^2K} + \frac{thK}{ch^2H} \cos \theta \right) + i \frac{thK}{ch^2H} \sin \theta \right), \\ &= y \frac{\left(sh(2H) + sh(2K) \cos \theta \right) + i sh(2K) \sin \theta}{2ch^2H ch^2K}.\end{aligned}$$

Therefore, the two solutions of (3.11) are:

$$(3.14) \quad \beta_{(1)} = \frac{m - \sqrt{m^2 - f_B(2H, 2K)}}{f_B(2H, 2K)} \cdot \left[\left(sh(2H) + sh(2K) \cos \theta \right) + i sh(2K) \right]$$

and

$$\beta_{(2)} = \frac{m + \sqrt{m^2 - f_B(2H, 2K)}}{f_B(2H, 2K)} \cdot \left[\left(sh(2H) + sh(2K) \cos \theta \right) + i sh(2K) \right].$$

Here we need to discuss whether both $\beta_{(1)}$ and $\beta_{(2)}$ are valid solutions. Using the properties of hyperbolic functions we show that

$$\begin{aligned}f_B(2H, 2K) &= sh^2(2H) + sh^2(2K) + 2sh(2H)sh(2K) \cos \theta \\ &< ch^2(2H) + ch^2(2K) + 2ch(2H)ch(2K) \\ &= \left(ch(2H) + ch(2K) \right)^2,\end{aligned}$$

and hence $f_B(2H, 2K) < m^2$.

Then

$$|\beta_{(2)}|^2 = \frac{2 \left(m^2 + m \sqrt{m^2 - f_B(2H, 2K)} \right) - f_B(2H, 2K)}{f_B(2H, 2K)}$$

$$\begin{aligned}
&= \frac{2\left(m^2 + m\sqrt{m^2 - f_B(2H, 2K)}\right)}{f_B(2H, 2K)} - 1 \\
&> \frac{2m^2}{f_B(2H, 2K)} - 1 \\
&> \frac{2m^2}{m^2} - 1 \\
&= 1,
\end{aligned}$$

which shows that the modulus of $\beta_{(2)}$ is greater than 1, and therefore $\beta_{(2)}$ does not belong to the unit disk. So, $\beta_{(1)}$ is the only valid expression for the centroid.

Let

$$\beta_0 = (sh(2H) + sh(2K)\cos\theta) + i sh(2K)\sin\theta.$$

Then

$$|\beta_0|^2 = sh^2(2H) + sh^2(2K) + 2sh(2H)sh(2K) = f_B(2H, 2K),$$

and now we can rewrite (3.14) as:

$$\beta = \frac{m - \sqrt{m^2 - |\beta_0|^2}}{|\beta_0|^2} \beta_0.$$

Using (1.14), we can write the the expression for β as follows:

$$(3.15) \quad \beta = \left[m - \sqrt{m^2 - |\beta_0|^2} \right] \beta_0^*.$$

3.3 Orthocenter

We start from the observing that if HKO is a right triangle, its orthocenter coincides with the origin, and hence we can exclude the right triangles from our further

discussion. Let $\gamma = \gamma_1 + \gamma_2$ be the orthocenter of a hyperbolic triangle. Using the same reasoning as in the cases with the circumcenter and the centroid, we find γ as a common point of the two altitudes KA_K and OA_O . So we need to solve simultaneously the two following equations:

$$\begin{pmatrix} \gamma & 1 \end{pmatrix} \begin{bmatrix} -2k \cos \theta & k^2 + 1 \\ k^2 + 1 & -2k \cos \theta \end{bmatrix} \begin{pmatrix} \bar{\gamma} \\ 1 \end{pmatrix} = 0$$

and

$$\begin{pmatrix} \gamma & 1 \end{pmatrix} \begin{bmatrix} 0 & (h^+ - k^+ \cos \theta) + ik^+ \sin \theta \\ (h^+ - k^+ \cos \theta) - ik^+ \sin \theta & 0 \end{bmatrix} \begin{pmatrix} \bar{\gamma} \\ 1 \end{pmatrix} = 0.$$

These two equations are equivalent to:

$$(3.16) \quad -2k \cos \theta (\gamma_1^2 + \gamma_2^2 + 1) + 2(k^2 + 1)\gamma_1 = 0$$

and

$$(h^+ - k^+ \cos \theta)\gamma_1 - k^+ \sin \theta \gamma_2 = 0.$$

From the last equation

$$\gamma_2 = \frac{(h^+ - k^+ \cos \theta)}{k^+ \sin \theta} \gamma_1$$

After the substitution this expression for γ_2 into (3.16), we obtain:

$$\begin{aligned} & \left(\gamma_1^2 + \gamma_1^2 \left(\frac{h^+ - k^+ \cos \theta}{k^+ \sin \theta} \right)^2 + 1 \right) k \cos \theta - (k^2 + 1)\gamma_1 = 0 \\ & \left(\frac{\gamma_1}{k^+ \sin \theta} \right)^2 \left((k^+ \sin \theta)^2 + (h^+)^2 + (k^+ \cos \theta)^2 - 2h^+ k^+ \cos \theta \right) k \cos \theta - \\ & - \frac{\gamma_1}{k^+ \sin \theta} (k^2 + 1)(k^+ \sin \theta) + k \cos \theta = 0. \end{aligned}$$

With the help of another substitution

$$x = \frac{\gamma_1}{k^+ \sin \theta}$$

and some simplification we will receive the quadratic equation:

$$(3.17) \quad x^2 \left((k^+)^2 + (h^+)^2 - 2h^+k^+ \cos \theta \right) - x(k^2 + 1)(h^2 + 1) \tan \theta + 1 = 0.$$

Let us now introduce two new variables p and q such that

$$p = \frac{2h}{1+h^2} \quad \text{and} \quad q = \frac{2k}{1+k^2}.$$

Hence

$$k^2 + 1 = \frac{2k}{q}; \quad h^2 + 1 = \frac{2h}{p}; \quad k^+ = \frac{2hk}{p}; \quad h^+ = \frac{2hk}{q}$$

Then we can rewrite (3.17) as follows:

$$x^2 \left(\left(\frac{2kh}{p} \right)^2 + \left(\frac{2kh}{q} \right)^2 - \frac{8k^2h^2}{pq} \cos \theta \right) - x \left(\frac{2k}{p} \frac{2h}{q} \tan \theta \right) + 1 = 0,$$

$$\left(x \frac{2kh}{pq} \right)^2 (q^2 + p^2 - 2pq \cos \theta) - 2 \left(x \frac{2kh}{pq} \right) \tan \theta + 1 = 0.$$

The solutions of this equation are:

$$x_{(1),(2)} = \frac{\tan \theta \pm \sqrt{\tan^2 \theta - (q^2 + p^2 - 2pq \cos \theta)}}{q^2 + p^2 - 2pq \cos \theta} \cdot \frac{pq}{2kh},$$

and thus the solutions for γ_1 :

$$\gamma_1 = \frac{\tan \theta \pm \sqrt{\tan^2 \theta - (q^2 + p^2 - 2pq \cos \theta)}}{q^2 + p^2 - 2pq \cos \theta} q \sin \theta.$$

We want to express γ_1 in non-Euclidean terms. But before we do that let us recall that

$$(3.18) \quad th(2H) = \frac{2thH}{1+th^2H} = \frac{2h}{1+h^2} = p \quad \text{and} \quad th(2K) = \frac{2thK}{1+th^2K} = \frac{2k}{1+k^2} = q,$$

and make one more substitution :

$$th^2(2H) + th^2(2K) - 2th(2H)th(2K)\cos\theta = f_\gamma(2H, 2K).$$

Now

$$\gamma_1 = \frac{\tan\theta \pm \sqrt{\tan^2\theta - f_\gamma(2H, 2K)}}{f_\gamma(2H, 2K)} th(2K) \sin\theta,$$

$$\gamma_2 = \frac{\tan\theta \pm \sqrt{\tan^2\theta - f_\gamma(2H, 2K)}}{f_\gamma(2H, 2K)} (th(2H) - th(2K)\cos\theta)$$

$$\gamma = \frac{\tan\theta \pm \sqrt{\tan^2\theta - f_\gamma(2H, 2K)}}{f_\gamma(2H, 2K)} (th(2K) \sin\theta + i(th(2H) - th(2K)\cos\theta)).$$

Let

$$\gamma_0 = th(2K) \sin\theta + i(th(2H) - th(2K)\cos\theta)$$

Then

$$|\gamma_0|^2 = f_\gamma(2H, 2K)$$

and

$$(3.19) \quad \gamma = \left[\tan\theta \pm \sqrt{\tan^2\theta - |\gamma_0|^2} \right] \gamma_0^*.$$

In order for γ belong to the unit disk, we should require $|\gamma| < 1$. If we perform some algebraic transformations, we find that if $\theta \in (0; \frac{\pi}{2})$, (3.19) we must choose the negative square root, and if $\theta \in (\frac{\pi}{2}; \pi)$ - positive.

CHAPTER FOUR

THE ANGLES OF PARALLELISM

4.1 How to Define the Kind of Pencil

Theorem 4.1

Let V and W be two circles orthogonal to the unit circle. Let also $v = v_1 + iv_2$ and $w = w_1 + iw_2$ be the centers of V and W respectively. Then the pencil of circles, generated by V and W is:

elliptic, if $(v_1w_2 - v_2w_1)^2 > (w_1 - v_1)^2 + (w_2 - v_2)^2$,

parabolic, if $(v_1w_2 - v_2w_1)^2 = (w_1 - v_1)^2 + (w_2 - v_2)^2$,

hyperbolic, otherwise.

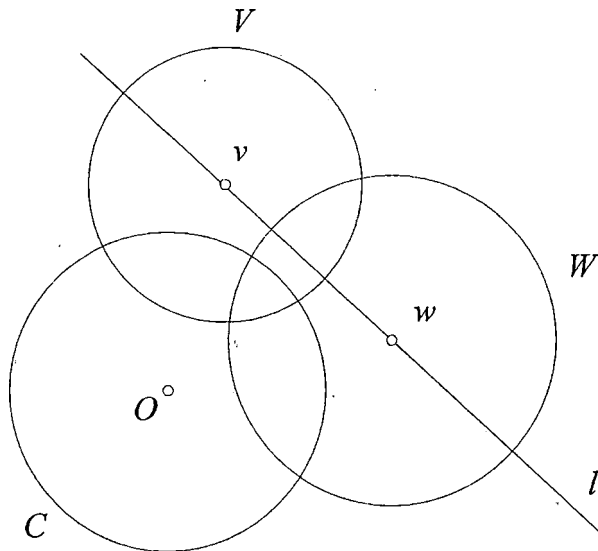


Figure 4.1

Proof. A line, say l , that goes through the points v and w , is the only straight line of the pencil defined by the circles V and W . We can define the kind of the pencil generated by V and W by the position of the line l with respect to the unit circle: if l is exterior to the unit circle, then the pencil is elliptic (as we have drawn on Figure (4.1)); if l is tangent to the unit circle, the pencil is parabolic; and finally, if l is interior to the unit circle, the pencil is hyperbolic.

Now, using the conjecture from (1.12), we can restate the statement above as follows: let φ be an angle between the straight line l and the unit circle. Then the pencil generated by V and W is

elliptic, if and only if $|\cos \varphi| > 1$,

parabolic, if and only if $|\cos \varphi| = 1$,

hyperbolic, if and only if $|\cos \varphi| < 1$.

Therefore our nearest goal is to express $\cos \varphi$ through v and w . In the formula (1.9) let $H_1 = H_U$ represent a unit circle, so by (1.7)

$$H_U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

and let also $H_2 = H$ be a straight line through v and w . Thus H is of the form

$$H = \begin{bmatrix} 0 & B \\ \bar{B} & D \end{bmatrix},$$

where $B = b_1 + ib_2$ as usual. Applying (1.9) and (1.10) to our case, we get

$$\text{Tr}(H_U, H) = D,$$

and thus

$$\cos \varphi = \frac{D}{-2\sqrt{|\det H|}}.$$

Now we need to express the last formula through the coordinates of the centers of the two given circles. Since our straight line is supposed to go through both v and w , its matrix has to satisfy the two conditions:

$$\begin{bmatrix} v & 1 \end{bmatrix} \begin{bmatrix} 0 & B \\ \bar{B} & D \end{bmatrix} \begin{bmatrix} \bar{v} \\ 1 \end{bmatrix} = 0$$

and

$$\begin{bmatrix} w & 1 \end{bmatrix} \begin{bmatrix} 0 & B \\ \bar{B} & D \end{bmatrix} \begin{bmatrix} \bar{w} \\ 1 \end{bmatrix} = 0.$$

This gives us two equations:

$$(4.1) \quad 2(b_1 v_1 - b_2 v_2) + D = 0 \quad \text{and} \quad 2(b_1 w_1 - b_2 w_2) + D = 0,$$

which we need to solve simultaneously. The solutions of (4.1) are:

$$B = b_1 \frac{(w_2 - v_2) + (w_1 - v_1)i}{w_2 - v_2}$$

and

$$D = 2b_1 \frac{v_2 w_1 - v_1 w_2}{w_2 - v_2}.$$

Thus Hermitian matrix H_l of the straight line l is:

$$H_l = \begin{bmatrix} 0 & b_1 \frac{(w_2 - v_2) + (w_1 - v_1)i}{w_2 - v_2} \\ b_1 \frac{(w_2 - v_2) - (w_1 - v_1)i}{w_2 - v_2} & 2b_1 \frac{v_2 w_1 - v_1 w_2}{w_2 - v_2} \end{bmatrix}.$$

Using (1.6), we can rewrite H_l as follows:

$$H_l = \begin{bmatrix} 0 & (w_2 - v_2) + (w_1 - v_1)i \\ (w_2 - v_2) - (w_1 - v_1)i & 2(v_2 w_1 - v_1 w_2) \end{bmatrix}.$$

The determinant of the last matrix is:

$$\det H_l = -((w_2 - v_2)^2 + (w_1 - v_1)^2).$$

Finally we find the angle φ between the straight line l and the unit circle as:

$$(4.2) \quad \cos \varphi = \frac{v_1 w_2 - v_2 w_1}{\sqrt{(w_1 - v_1)^2 + (w_2 - v_2)^2}}$$

It then follows that the pencil is elliptic, if and only if

$$\left| \frac{v_1 w_2 - v_2 w_1}{\sqrt{(w_1 - v_1)^2 + (w_2 - v_2)^2}} \right| > 1,$$

parabolic, if and only if

$$\left| \frac{v_1 w_2 - v_2 w_1}{\sqrt{(w_1 - v_1)^2 + (w_2 - v_2)^2}} \right| = 1,$$

and hyperbolic, if and only if

$$\left| \frac{v_1 w_2 - v_2 w_1}{\sqrt{(w_1 - v_1)^2 + (w_2 - v_2)^2}} \right| < 1,$$

which proves Theorem 4.1.

4.2 Circumparallelism

Definition 4.1

Let us consider a set P_{HK} of hyperbolic triangles HKO with the sides KO and HO of fixed length H and K respectively, and the angle $HOK = \theta$, where $0 < \theta < \pi$. The angle θ_p correspondent to the $\triangle HKO \in P_{HK}$, whose perpendicular bisectors are mutually parallel we will call the angle of circumparallelism. In Figure (4.2) we demonstrated $\triangle HKO$ whose angle θ is the angle of circumparallelism.

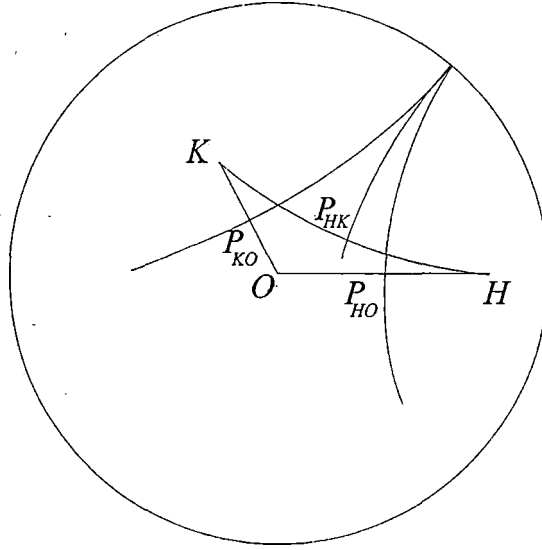


Figure 4.2

It was proved earlier that the circles that afford the three perpendicular bisectors of the sides of a triangle belong to the same pencil. The point of their concurrence exists and belongs to the unit disk only in the case when the pencil is elliptic, thus we can not claim that the circumcenter of the hyperbolic triangle always exists. To explore the conditions of its existence, according to Theorem 4.1, we need to find the angle φ_p between the only

straight line of the pencil and the unit circle. Consider the circles KO and HO to be generators of the pencil of perpendicular bisectors. Let the centers of KO and HO be v and w respectively. Then $v = \frac{e^{i\theta}}{k} = \frac{1}{k} \cos \theta + i \frac{1}{k} \sin \theta$ and $w = \frac{1}{h}$. Substituting v and w into (4.2), and performing some algebraic transformations, we obtain:

$$\cos \varphi_p = \frac{-\sin \theta}{\sqrt{h^2 + k^2 - 2hk \cos \theta}} = \frac{-\sin \theta}{\sqrt{th^2 H + th^2 K - 2thHthK \cos \theta}} = \frac{-\sin \theta}{|\alpha_0|}.$$

Hence the perpendicular bisectors are parallel if

$$\frac{\sin^2 \theta}{th^2 H + th^2 K - 2thHthK \cos \theta} = 1,$$

Given that $\sin^2 \theta = 1 - \cos^2 \theta$, the previous equation is equivalent to the quadratic equation

$$\cos^2 \theta - 2thHthK \cos \theta + (th^2 H + th^2 K - 1) = 0,$$

which yields

$$\cos \theta = thHthK \pm \sqrt{(1 - th^2 H)(1 - th^2 K)},$$

$$(4.3) \quad \cos \theta = thHthK \pm schHschK.$$

Let us notice here that since we agreed that $H > K$,

$$-1 < thHthK - schHschK < 0 \quad \text{and} \quad 0 < thHthK + schHschK < 1,$$

and so if the angle θ of the triangle HKO is acute, its angle of circumparallelism is

$$\cos \theta_p = thHthK - schHschK,$$

while if the angle θ is obtuse, the angle of circumparallelism is

$$\cos \theta_p = thHthK + schHschK.$$

It follows, that the pencil of the perpendicular bisectors is elliptic if and only if

$$(4.4) \quad thHthK - schHschK < \cos \theta < thHthK + schHschK$$

In the case of an isosceles triangle,

$$\cos \theta = th^2H \pm sch^2H,$$

and since the case $\cos \theta = th^2H + sch^2H = 1$ is impossible for the triangle, we are left with only one angle of circumparallelism;

$$\cos \theta_p = th^2H - sch^2H,$$

which simplifies to

$$(4.5) \quad \cos \theta_p = 2th^2H - 1.$$

Another way to express (4.5) is:

$$\cos \frac{\theta_p}{2} = thH,$$

In this case the pencil is elliptic if and only if

$$(4.6) \quad \cos \theta > 2th^2H - 1.$$

The conditions (4.4) and (4.6) are also the conditions of the existence of the circumcenter of a hyperbolic triangle.

4.3 Orthoparallelism

Definition 4.2.

Let us consider the set P_{HK} of hyperbolic triangles, that we defined above. The angle θ_a corresponding to the $\triangle HKO \in P_{HK}$, whose altitudes are mutually parallel we

will call the angle of orthoparallelism. In Figure (4.3) three altitudes of $\triangle HKO$ meet on the boundary of the unit circle and thus mutually parallel.

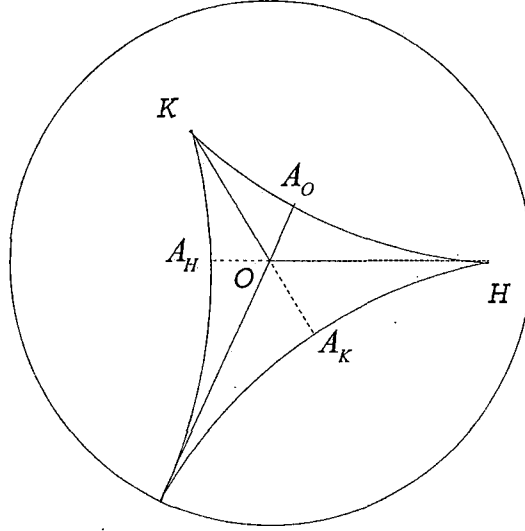


Figure 4.3

Let φ_a denote the angle between the only straight line of the pencil of the altitudes and the unit circle. Using arguments similar to those in the case with the perpendicular bisectors, we calculate $\cos \varphi_a$:

$$|\cos \varphi_a| = \frac{h^+ k^+ \sin \theta}{2hk \cos \theta \sqrt{h^+ + k^+ - 2hk \cos \theta}}.$$

The condition for the altitudes to be parallel is $|\cos \varphi_a| = 1$ or $\cos^2 \varphi_a = 1$.

Squaring the last equation and applying the substitutions (3.18) from the Chapter III, we obtain the following equation:

$$pq\left(\frac{1}{p^2} + \frac{1}{q^2} - \frac{2}{pq} \cos \theta\right) - \frac{1}{\cos^2 \theta} + 1 = 0,$$

which yields the cubic equation

$$(4.7) \quad 2pq \cos^3 \theta - (p^2 + q^2 + 1) \cos^2 \theta + 1 = 0.$$

The last equation has three real roots, but only two of them have modulus less than one:

$$\cos \theta_{(1)} \in (-1, 0)$$

$$\cos \theta_{(2)} \in (0, 1),$$

and thus we have two angles of orthoparallelism - one acute, and one obtuse.

In the case of an isosceles triangle (4.7) becomes

$$2p^2 \cos^3 \theta - (2p^2 + 1) \cos^2 \theta + 1 = 0,$$

$$(\cos \theta - 1)(2p^2 \cos^2 \theta - \cos \theta - 1) = 0.$$

Since $\cos \theta \neq 1$ for any $\theta \in (0, \pi)$, the only possibility we have is

$$2p^2 \cos^2 \theta - \cos \theta - 1 = 0,$$

solutions of which are:

$$\cos \theta = \frac{1 \pm \sqrt{1 + 8p^2}}{4p^2}.$$

Let us evaluate the expression $\frac{1 + \sqrt{1 + 8p^2}}{4p^2}$. Since $p = th(2H) < 1$,

$$\frac{1 + \sqrt{1 + 8p^2}}{4p^2} > \frac{p + \sqrt{p^2 + 8p^2}}{4p^2} = \frac{1}{p} > 1$$

Since we need $|\cos \theta| < 1$, in the isosceles case there is one angle of orthoparallelism:

$$\cos \theta_a = \frac{1 - \sqrt{1 + 8p^2}}{4p^2},$$

or after we substitute $p = th(2H)$,

$$(4.8) \quad \cos \theta_a = \frac{1 - \sqrt{1 + 8th^2(2H)}}{4th^2(2H)}.$$

The function $\cos \theta_a(th(2H))$ does not have critical points, and therefore it varies monotonically with H . It follows that the pencil of the altitudes is elliptic, if $0 < \theta < \theta_a$, or

$$(4.9) \quad \cos \theta > \frac{1 - \sqrt{1 + 8th^2(2H)}}{4th^2(2H)}$$

The next question we want to discuss – is there any isosceles triangle such that its angle of circumparallelism is also the angle of orthoparallelism? It follows from (4.5) and (4.8), that this is true if

$$\frac{1 - \sqrt{1 + 8th^2(2H)}}{4th^2(2H)} = 2th^2H - 1.$$

The solution of the last equation is

$$th(2H) = \frac{\sqrt{5}}{3},$$

and thus there is a unique hyperbolic triangle, that has the required property. Its angle θ of ortho- circumparallelism is obtuse and $\cos \theta = -\frac{3}{5}$, the length of the equal sides is

$H = th^{-1} \frac{\sqrt{5}}{5}$, and the base HK is of the length $HK = th^{-1} \frac{\sqrt{2}}{2}$. In Figure (4.4) we have

drawn $\triangle HKO$ whose angle of circumparallelism is also the angle of orthoparallelism.

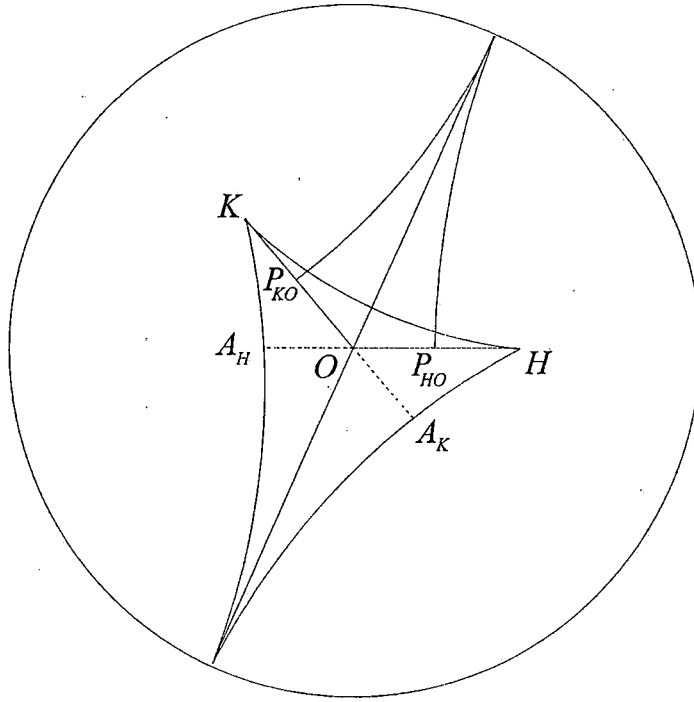


Figure 4.4

Finally, combining (4.6) and (4.9), we obtain

$$\cos \theta > \max [\theta_a, \theta_p] = \begin{cases} \theta_a, & \text{if } h < \frac{\sqrt{5}}{5} \\ \theta_p, & \text{if } h > \frac{\sqrt{5}}{5} \end{cases}.$$

which states the conditions that the angle θ of the isosceles hyperbolic triangle needs to satisfy in order for both the circumcenter and the orthocenter exist.

CHAPTER FIVE

EULER LINE AND ITS PROPERTIES

5.1 Euler Line

Definition 5.1.

A hyperbolic Euler line is a d -line that goes through the circumcenter and the centroid of a nonequilateral hyperbolic triangle. We should add two remarks to this definition:

- 1) The Euler line of a hyperbolic triangle exists if and only if there exist the circumcenter and the centroid.
- 2) In an equilateral hyperbolic triangle (as in an equilateral Euclidean triangle) the circumcenter and the centroid coincide (the proof of this property can be found in a number of books on non-Euclidean geometry), and define a point-circle instead of a d -line.

Let us recall one of the expressions of the hyperbolic circumcenter that we obtained in Chapter III :

$$\alpha = \frac{\left[\sin \theta - \sqrt{\sin^2 \theta - |\alpha_0|^2} \right]}{|\alpha_0|^2} \alpha_0,$$

where $\alpha_0 = thH \sin \theta + i(thK - thH \cos \theta)$. To simplify the notations in our further calculations, let us introduce the following substitutions:

$$(5.1) \quad \alpha_{10} = \left[\sin \theta - \sqrt{\sin^2 \theta - |\alpha_0|^2} \right],$$

$$\alpha_{01} = thH \sin \theta \quad \text{and} \quad \alpha_{02} = thK - thH \cos \theta.$$

Then $\alpha_0 = \alpha_{01} + i\alpha_{02}$ and

$$(5.2) \quad \alpha = \frac{\alpha_{10}}{|\alpha_0|^2}(\alpha_{01} + i\alpha_{02})$$

$$\alpha = \frac{\alpha_{10}}{|\alpha_0|^2}(\alpha_{01} + i\alpha_{02})$$

Now, in the expression of the centroid:

$$\beta = \frac{m - \sqrt{m^2 - |\beta_0|^2}}{\beta_0^2} \beta_0,$$

where $\beta_0 = (sh(2H) + sh(2K) \cos \theta) + i sh(2K) \sin \theta$, let

$$(5.3) \quad \beta_{10} = \left[m - \sqrt{m^2 - |\beta_0|^2} \right],$$

$$\beta_{01} = sh(2H) + sh(2K) \cos \theta \quad \text{and} \quad \beta_{02} = sh(2K) \sin \theta.$$

It then follows that $\beta_0 = \beta_{01} + i\beta_{02}$ and

$$(5.4) \quad \beta = \frac{\beta_{10}}{|\beta_0|^2}(\beta_{01} + i\beta_{02}).$$

For the algebraic representation of the Euler line, let us use the Hermitian matrix H_E of the form:

$$(5.5) \quad H_E = \begin{bmatrix} 1 & B \\ \bar{B} & 1 \end{bmatrix},$$

where $B = b_1 + ib_2$. It follows from (5.5) and the definition of the Euler line, that

$$(5.6) \quad \begin{pmatrix} \alpha & 1 \end{pmatrix} \begin{bmatrix} 1 & B \\ \bar{B} & 1 \end{bmatrix} \begin{pmatrix} \bar{\alpha} \\ 1 \end{pmatrix} = 0,$$

$$(5.7) \quad \begin{pmatrix} \beta & 1 \end{pmatrix} \begin{bmatrix} 1 & B \\ \overline{B} & 1 \end{bmatrix} \begin{pmatrix} \overline{\beta} \\ 1 \end{pmatrix} = 0.$$

Then from the equation (5.6) we will have:

$$(\alpha_1^2 + \alpha_2^2 + 1) + 2(b_1\alpha_1 - b_2\alpha_2) = 0,$$

and if we now recall that $\alpha_1^2 + \alpha_2^2 = |\alpha|^2$, we can rewrite the last equation as follows:

$$(5.8) \quad |\alpha|^2 + 2(b_1\alpha_1 - b_2\alpha_2) + 1 = 0.$$

According to (5.2), $|\alpha|^2 = \frac{\alpha_{10}^2}{|\alpha_0|^2}$ and

$$b_1\alpha_1 - b_2\alpha_2 = \frac{\alpha_{10}}{|\alpha_0|}(b_1\alpha_{01} - b_2\alpha_{02}).$$

Applying this information and observing that $|\alpha_0| \neq 0$, from the equation (5.8) we obtain:

$$\alpha_{10}^2 + 2\alpha_{10}(\alpha_{01}b_1 - \alpha_{02}b_2) + |\alpha_0|^2 = 0,$$

which yields the expression for b_2 :

$$(5.9) \quad b_2 = \frac{\alpha_{10}^2 + |\alpha_0|^2 + 2\alpha_{10}\alpha_{01}b_1}{2\alpha_{10}\alpha_{02}}$$

Now, using arguments similar to the case with the circumcenter, and applying (5.4) to (5.7), we obtain

$$\beta_{10}^2 + 2\beta_{10}(\beta_{01}b_1 - \beta_{02}b_2) + |\beta_0|^2 = 0.$$

By substituting the value of b_2 from (5.9) into this equation we receive:

$$\beta_{10}^2\alpha_{10}\alpha_{02} + 2\beta_{10}\beta_{01}b_1\alpha_{10}\alpha_{02} - \beta_{10}\beta_{02}(\alpha_{10}^2 + |\alpha_0|^2 + 2\alpha_{10}\alpha_{01}b_1) + |\beta_0|^2\alpha_{10}\alpha_{02} = 0.$$

Therefore

$$(5.10) \quad b_1 = \frac{\beta_{10}\beta_{02}(\alpha_{10}^2 + |\alpha_0|^2) - \alpha_{10}\alpha_{02}(\beta_{10}^2 + |\beta_0|^2)}{2(\beta_{10}\beta_{01}\alpha_{10}\alpha_{02} - \alpha_{10}\alpha_{01}\beta_{10}\beta_{02})}$$

According to (5.1) $\alpha_{10}^2 = 2\sin^2\theta - |\alpha_0|^2 - 2\sin\theta\sqrt{\sin^2\theta - |\alpha_0|^2}$,

$$\alpha_{10}^2 + |\alpha_0|^2 = 2\alpha_{10}\sin\theta.$$

It follows from (5.3) that

$$\beta_{10}^2 = 2m\beta_{10} - |\beta_0|^2.$$

$$\beta_{10}^2 + |\beta_0|^2 = 2m\beta_{10}$$

Now, we can rewrite (5.10) as follows:

$$b_1 = \frac{\beta_{02}\sin\theta - \alpha_{02}m}{\beta_{01}\alpha_{02} - \alpha_{01}\beta_{02}}.$$

Then from (5.9) and (5.10) we find b_2 :

$$b_2 = \frac{\sin\theta + \alpha_{01}b_1}{\alpha_{02}} = \frac{\beta_{01}\sin\theta - \alpha_{01}m}{\beta_{01}\alpha_{02} - \alpha_{01}\beta_{02}}.$$

Let

$$(5.11) \quad \Delta = \beta_{01}\alpha_{02} - \alpha_{01}\beta_{02}.$$

(It is interesting to notice that $\beta_{01}\alpha_{02} - \alpha_{01}\beta_{02} = \Delta = \text{Im}(\alpha_0\bar{\beta}_0)$.) With the last

substitution made,

$$\begin{aligned} B &= \frac{1}{\Delta}((\beta_{02}\sin\theta - \alpha_{02}m) + i(\beta_{01}\sin\theta - \alpha_{01}m)) \\ &= \frac{1}{\Delta}\sin\theta(\beta_{02} + i\beta_{01}) - m(\alpha_{02} + i\alpha_{01}) \\ &= \frac{(\beta_{02}\sin\theta - \alpha_{02}m) + i(\beta_{01}\sin\theta - \alpha_{01}m)}{\Delta} \end{aligned}$$

Finally, the Hermitian representation H_E for the hyperbolic Euler line is found:

$$H_E = \begin{bmatrix} 1 & \frac{(\beta_{02} \sin \theta - \alpha_{02} m) + i(\beta_{01} \sin \theta - \alpha_{01} m)}{\Delta} \\ \frac{(\beta_{02} \sin \theta - \alpha_{02} m) - i(\beta_{01} \sin \theta - \alpha_{01} m)}{\Delta} & 1 \end{bmatrix}$$

$$(5.12) \quad H_E =$$

$$\begin{bmatrix} \Delta & (\beta_{02} \sin \theta - \alpha_{02} m) + i(\beta_{01} \sin \theta - \alpha_{01} m) \\ (\beta_{02} \sin \theta - \alpha_{02} m) - i(\beta_{01} \sin \theta - \alpha_{01} m) & \Delta \end{bmatrix},$$

We could simplify the notation if we notice that

$$\begin{aligned} B &= \frac{1}{\Delta} \sin \theta (\beta_{02} + i\beta_{01}) - m(\alpha_{02} + i\alpha_{01}) \\ &= \frac{-i}{\Delta} (\sin \theta (\beta_{01} - i\beta_{02}) - m(\alpha_{01} - i\alpha_{02})) \\ &= \frac{i}{\Delta} (m\bar{\alpha}_0 - \bar{\beta}_0 \sin \theta). \end{aligned}$$

The last observation gives us another form of H_E :

$$H_E = \begin{bmatrix} \Delta & i(m\bar{\alpha}_0 - \bar{\beta}_0 \sin \theta) \\ -i(m\alpha_0 - \beta_0 \sin \theta) & \Delta \end{bmatrix},$$

where $\Delta = \text{Im}(\alpha_0 \bar{\beta}_0)$.

As a verification of our expression, let us find Hermitian representation of the Euler line in the isosceles triangle. Consider the nonequilateral triangle HKO such that $HO = KO = H$. To compute Δ and $B' = \Delta \cdot B$, we use (5.11) and (5.12) :

$$\Delta = sh(2H)(1 + \cos \theta)thH(1 - \cos \theta) - sh(2H)thH \sin^2 \theta = 0.$$

$$B' = (\beta_{02} \sin \theta - \alpha_{02} m) + i(\beta_{01} \sin \theta - \alpha_{01} m).$$

$$\begin{aligned} \operatorname{Re}(B') &= sh(2H) \sin^2 \theta - thH(1 - \cos \theta)(2ch(2H) + 1) \\ &= thH(2ch^2H \sin^2 \theta - (1 - \cos \theta)(2sh^2H + 2ch^2H + 1)) \\ &= thH(2ch^2H(\sin^2 \theta - (1 - \cos \theta)) - (1 - \cos \theta)(2sh^2H + 1)) \\ &= thH(2ch^2H \cos \theta - 2sh^2H - 1)(1 - \cos \theta) \\ &= [thH(2ch^2H \cos \theta - 2sh^2H - 1)] 2 \sin^2 \frac{\theta}{2}. \end{aligned}$$

$$\begin{aligned} \operatorname{Im}(B') &= sh(2H)(1 + \cos \theta) \sin \theta - thH \sin \theta (2ch(2H) + 1) \\ &= thH \sin \theta (2ch^2H(1 + \cos \theta) - (2sh^2H + 2ch^2H + 1)) \\ &= thH(2ch^2H \cos \theta - 2sh^2H - 1) \sin \theta \\ &= [thH(2ch^2H \cos \theta - 2sh^2H - 1)] 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}. \end{aligned}$$

$$B' = \left[2thH(2ch^2H \cos \theta - 2sh^2H - 1) \sin \frac{\theta}{2} \right] \left(\sin \frac{\theta}{2} + i \cos \frac{\theta}{2} \right).$$

Finally, substituting the values for Δ and B' into (5.12) and multiplying the matrix by $\left[2thH(2ch^2H \cos \theta - 2sh^2H - 1) \sin \frac{\theta}{2} \right]^{-1}$, we obtain:

$$H_E = \begin{bmatrix} 0 & \sin \frac{\theta}{2} + i \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} - i \cos \frac{\theta}{2} & 0 \end{bmatrix}.$$

The last matrix represents a bisector of the angle θ , which makes sense since in an isosceles (hyperbolic or Euclidean) triangle HKO with the base HK , the bisector and the

median of the angle O as well as the perpendicular bisector of HK is the same line, which is also the Euler line of the triangle.

5.2 When does the Orthocenter Belong to the Euler Line

In Euclidean geometry the orthocenter of a triangle belongs to the Euler line. An elegant proof of this statement can be found in [2]. Let us just point out that the proof uses the relationship of similarity of the triangles. Things are different and much more complicated in non-Euclidean geometry, where there is no distinction between similarity and d-congruency of the triangles, and so the idea of the Euclidean proof can not be used. (“In non-Euclidean geometry two d-figures are d-congruent if there is a non-Euclidean transformation that maps one onto the other”. [1])

Theorem 5.1

If the orthocenter of an isosceles hyperbolic triangle exists then it belongs to the Euler line.

Proof. Let HKO be an isosceles triangle with $HO = KO$. Then the perpendicular bisector of the side HK is also the median and the altitude. Therefore all three points: circumcenter, centroid and orthocenter belong to the same d -line, and this d -line is the hyperbolic Euler line of the triangle.

Theorem 5.1 tells us that it is possible for the orthocenter of a hyperbolic triangle to belong to the Euler line. Although we still do not know, if it is possible for a scalene hyperbolic triangle to have the named property. To continue our investigation of the question, let us recall the expression for the orthocenter of a triangle that we found in Chapter III:

$$\gamma = \frac{\left[\tan \theta - \sqrt{\tan^2 \theta - |\gamma_0|^2} \right]}{|\gamma_0|^2} \gamma_0,$$

where $\gamma_0 = th(2K) \sin \theta + i(th(2H) - th(2K) \cos \theta)$. As in the case of the circumcenter and the centroid, we want to make same substitutions. We set

$$\gamma_{01} = \left[\tan \theta - \sqrt{\tan^2 \theta - |\gamma_0|^2} \right],$$

$$\gamma_{01} = th(2K) \sin \theta \quad \text{and} \quad \gamma_{02} = th(2H) - th(2K) \cos \theta.$$

Then $\gamma_0 = \gamma_{01} + i\gamma_{02}$ and

$$\gamma = \frac{\gamma_{10}}{|\gamma_0|^2} (\gamma_{01} + i\gamma_{02}).$$

In order for γ to belong to the Euler line, its coordinates have to satisfy the equation $H_E(\gamma) = 0$, or

$$(5.13) \quad \begin{pmatrix} \gamma & 1 \end{pmatrix} \begin{bmatrix} 1 & B \\ \overline{B} & 1 \end{bmatrix} \begin{pmatrix} \overline{\gamma} \\ 1 \end{pmatrix} = 0.$$

Let us look at the matrix (5.12) that represents the Euler line and then rewrite (5.13) as follows:

$$(5.14) \quad \begin{pmatrix} \gamma & 1 \end{pmatrix} \begin{bmatrix} \Delta & B' \\ \overline{B}' & \Delta \end{bmatrix} \begin{pmatrix} \overline{\gamma} \\ 1 \end{pmatrix} = 0,$$

where

$$B' = b'_1 + ib'_2 = (\beta_{02} \sin \theta - \alpha_{02} m) + i(\beta_{01} \sin \theta - \alpha_{01} m)$$

Performing some algebraic transformations of (5.14) we receive:

$$(|\gamma|^2 + 1)\Delta + 2\frac{\gamma_{10}}{|\gamma_0|^2}(\gamma_{01}b_1 - \gamma_{02}b_2) = 0.$$

Since $|\gamma|^2 = \frac{\gamma_{10}}{|\gamma_0|^2}$ and $\gamma_{10}^2 + |\gamma_0|^2 = 2\gamma_{10}\tan\theta$, from the equation above we obtain

$$\Delta \tan\theta + \gamma_{01}b'_1 - \gamma_{02}b'_2 = 0.$$

Substituting Δ , γ_{01} , γ_{02} , b'_1 and b'_2 for their values, we rewrite the last equation as follows:

$$\begin{aligned} &(\beta_{01}\alpha_{02} - \alpha_{01}\beta_{02})\tan\theta + (\beta_{02}\sin\theta - \alpha_{02}m)th\,2K\sin\theta - \\ & - (\beta_{01}\sin\theta - \alpha_{01}m)(th\,2H - th\,2K\cos\theta) = 0 \end{aligned}$$

The previous equation is equivalent to

$$\begin{aligned} (5.15) \quad &[(sh(2H) + sh(2K)\cos\theta)(thK - thH\cos\theta) - thH\,sh(2K)(1 - \cos^2\theta)] + \\ & + [sh(2K)(1 - \cos^2\theta) - (thK - thH\cos\theta)m]th(2K)\cos\theta - \\ & - [(sh(2H) + sh(2K)\cos\theta) - thH\,m](th(2H) - th(2K)\cos\theta)\cos\theta = 0. \end{aligned}$$

Now we need to rewrite the expression on the left hand side of (5.15) as a polynomial on $\cos\theta$, and the equation as $P(\cos\theta) = 0$, where

$$(5.16) \quad P(\cos\theta) = P_3\cos^3\theta + P_2\cos^2\theta + P_1\cos\theta + P_0$$

We will calculate each coefficient separately:

$$P_3 = -sh(2K)th(2K) + sh(2K)th(2K) = 0,$$

$$P_2 = sh(2H)th(2K) - sh(2K)th(2H)$$

$$= \frac{sh(2H)ch(2H)sh(2K) - sh(2K)sh(2H)ch(2K)}{ch(2H)ch(2K)}$$

$$P_2 = th(2H)th(2K)(ch(2H) - ch(2K)),$$

$$\begin{aligned} P_1 &= [sh(2K)thK - sh(2H)thH] + [(thHth(2H) - thKth(2K))m] + \\ &\quad + [sh(2K)th(2K) - sh(2H)th(2H)] \\ &= -[ch(2H) - ch(2K)] + \left[\frac{(ch(2H) + ch(2K) + 1)(ch(2H) - ch(2K))}{ch(2H)ch(2K)} \right] - \\ &\quad - \left[\frac{(ch(2H)ch(2K) + 1)(ch(2H) - ch(2K))}{ch(2H)ch(2K)} \right] \\ &= \frac{(ch(2H) + ch(2K) - 2ch(2H)ch(2K))(ch(2H) - ch(2K))}{ch(2H)ch(2K)} \end{aligned}$$

$$P_0 = sh(2H)thK - sh(2K)thH = thHthK(ch(2H) - ch(2K)).$$

Now, that all the coefficients are known, we can substitute them into (5.16).

(Before we do that, let us notice that each of the coefficients has a factor

$ch(2H) - ch(2K)$). We obtain:

$$\begin{aligned} P(\cos \theta) &= [ch(2H) - ch(2K)][th(2H)th(2K) \cos^2 \theta + \\ &\quad + \frac{(ch(2H) + ch(2K) - 2ch(2H)ch(2K))}{ch(2H)ch(2K)} \cos \theta + thHthK], \end{aligned}$$

and therefore $P(\cos \theta) = 0$ can be written as follows:

$$\begin{aligned} &[ch(2H) - ch(2K)][sh(2H)sh(2K) \cos^2 \theta + \\ &+ (ch(2H) + ch(2K) - 2ch(2H)ch(2K)) \cos \theta + \\ &+ thHthKch(2H)ch(2K)] = 0. \end{aligned}$$

The previous equation is equivalent to the union of two equations:

$$(5.17) \quad [ch(2H) - ch(2K)] = 0$$

and

$$(5.18) \quad sh(2H)sh(2K)\cos^2\theta + (ch(2H) + ch(2K) - 2ch(2H)ch(2K))\cos\theta + \\ + thHthKch(2H)ch(2K) = 0$$

From (5.17) it immediately follows that $H = K$. This result does not give us anything new, it was already stated in Theorem 5.1.

Let us now transfer our attention to (5.18), which is the quadratic equation. We first find the discriminant D :

$$D = (ch(2H) + ch(2K) - 2ch(2H)ch(2K))^2 - 4sh(2H)sh(2K)thHthKch(2H)ch(2K) \\ = (ch(2H) - ch(2K))^2,$$

and thus there are two cases to consider.

Case 1:

$$\cos\theta = \frac{ch(2H)ch(2K) - ch(2K)}{sh(2H)sh(2K)} = \frac{ch(2H) - 1}{sh(2H)} \cdot \frac{ch(2K)}{sh(2K)} = thHcth(2K),$$

Let us recall the expression (2.5) for $\cos H$ from Chapter II. By substituting

$\cos\theta = thHcth(2K)$ into (2.5), we find that

$$\cos H = thHcth(2K) = \cos\theta,$$

and thus we are dealing with the isosceles triangle again, only this time the equal sides are HK and KO .

Case 2:

$$\cos \theta = \frac{ch(2H)ch(2K) - ch(2H)}{sh(2H)sh(2K)} = \frac{ch(2K) - 1}{sh(2K)} \cdot \frac{ch(2H)}{sh(2H)} = thKcth(2H).$$

From (2.6) it follows that when $\cos \theta = thKcth(2H)$,

$$\cos K = thKcth(2H) = \cos \theta,$$

and this time the triangle HKO is isosceles with $HK = HO$.

Now all the possibilities are exhausted, and we can make a conclusion that will complete our discussion about the named property of the Euler line.

Theorem 5.2.

Assume that the orthocenter and Euler line of a hyperbolic triangle exist. Then the hyperbolic Euler line goes through the orthocenter of the triangle if and only if the triangle is isosceles.

Analytically we can state Theorem 5.2 as follows:

If hyperbolic triangle HKO is such that $h = k$ and θ is the vertex angle then the orthocenter belongs to the Euler line if and only if

$$\cos \theta > \max [\theta_a, \theta_p] = \begin{cases} \theta_a, & \text{if } h < \frac{\sqrt{5}}{5} \\ \theta_p, & \text{if } h > \frac{\sqrt{5}}{5} \end{cases}$$

CHAPTER SIX

SUMMARY

Let us consider Euclidean triangle $HOK_{(E)}$ such that $HO = h$, $KO = k$, $\angle HOK = \theta$, and non-Euclidean triangle $HOK_{(N)}$ (as in Chapter II), whose vertices coincide with the vertices of $HOK_{(E)}$. Let also $HK_{(E)} = g$. Using Cosine Theorem, we can express g in h , k and θ :

$$g^2 = h^2 + k^2 - 2hk \cos \theta.$$

We can also express g in non-Euclidean terms:

$$(6.1) \quad g^2 = th^2H + th^2K - 2thHthK \cos \theta.$$

Let us now consider Euclidean triangle $POQ_{(E)}$ such that $PO = p$, $QO = q$, $\angle POQ = \theta$, and non-Euclidean triangle $POQ_{(N)}$ such that its vertices coincide with the vertices of $POQ_{(E)}$ and $PO = 2H$, $QO = 2K$. Let also $PQ_{(E)} = s$. Using Cosine Theorem, we can express s in p , q and θ :

$$s^2 = p^2 + q^2 - 2pq \cos \theta$$

Being expressed in non-Euclidean terms, s will look as follows:

$$(6.2) \quad s^2 = th^22H + th^22K - 2th2Hth2K \cos \theta$$

The perpendicular bisectors of the sides of a hyperbolic triangle belong to the same pencil of circles. The circumcenter of the triangle exists if and only if this pencil is elliptic, which happens if

$$(6.3) \quad \sin^2 \theta > |\alpha_0|^2,$$

where

$$\alpha_0 = thH \sin \theta + i(thK - thH \cos \theta).$$

According to (6.1) we can rewrite the condition (6.3) in Euclidean terms:

$$\sin^2 \theta > g^2.$$

When the circumcenter exists it can be found as follows:

$$(6.4) \quad \alpha = \left[\sin \theta - \sqrt{\sin^2 \theta - |\alpha_0|^2} \right] \alpha_0^*,$$

and the radius R of the circumcircle is

$$(6.5) \quad R = th^{-1} \left(\left[\sin \theta - \sqrt{\sin^2 \theta - |\alpha_0|^2} \right] |\alpha_0^*| \right).$$

The formulas (6.4) and (6.5) above do not work for an isosceles right triangle, in which case

$$\alpha = \frac{thH}{2} (1 + i),$$

$$R = th^{-1} \left(\frac{\sqrt{2}}{2} thH \right).$$

The three medians of a hyperbolic triangle are always concurrent. We calculated the point of their concurrence β :

$$\beta = \left[m - \sqrt{m^2 - |\beta_0|^2} \right] \beta_0^*,$$

where

$$\beta_0 = (sh(2H) + sh(2K) \cos \theta) + ish(2K) \sin \theta,$$

and called β the centroid of a hyperbolic triangle by analogy with Euclidean geometry, although we never checked whether the point of concurrence of the medians is really the

centroid of the hyperbolic triangle. If in $\triangle HOK$ we connect point β with its vertices, we obtain three hyperbolic triangles. We will say that β is the centroid of the hyperbolic triangle if the areas of the triangles $HO\beta$, $KO\beta$, and $HK\beta$ are equal.

The three altitudes of a hyperbolic triangle belong to the same pencil of circles, which is elliptic if

$$2pq \cos^3 \theta - (p^2 + q^2 + 1) \cos^2 \theta + 1 > 0.$$

This condition is equivalent to the following :

$$(6.6) \quad \tan^2 \theta > |\gamma_0|^2,$$

where

$$\gamma_0 = th(2K) \sin \theta + i(th(2H) - th(2K) \cos \theta).$$

It follows from (6.2) that we can right (6.6) as:

$$\tan^2 \theta > s^2.$$

In the case when the pencil is elliptic, the altitudes are concurrent at the orthocenter γ of the triangle, which we can find as follows:

$$\gamma = \left[\tan \theta \pm \sqrt{\tan^2 \theta - |\gamma_0|^2} \right] \gamma_0^*,$$

choosing the negative square root for the acute θ and positive for the obtuse (in the right triangle $\gamma = 0$).

The hyperbolic Euler line exists if and only if the condition (6.3) holds. If the Euler line exists, it can be represented by the Hermitian matrix H_E :

$$H_E = \begin{bmatrix} \operatorname{Im}(\alpha_0 \bar{\beta}_0) & i(m\bar{\alpha}_0 - \bar{\beta}_0 \sin \theta) \\ -i(m\alpha_0 - \beta_0 \sin \theta) & \operatorname{Im}(\alpha_0 \bar{\beta}_0) \end{bmatrix}.$$

If θ is such that

$$1 - \cos^4 \theta > |\alpha_0|^2 + |\gamma_0|^2,$$

then both the circumcenter and the orthocenter exist.

As a result of this study we conclude the following theorem about the property of the Euler line:

Assume that the orthocenter and Euler line of a hyperbolic triangle exist. Then the hyperbolic Euler line goes through the orthocenter of the triangle if and only if the triangle is isosceles.

Analytically we can state this theorem as follows:

If hyperbolic triangle HKO is such that $h = k$ and θ is the vertex angle then the orthocenter belongs to the Euler line if and only if

$$\cos \theta > \max [\theta_a, \theta_p] = \begin{cases} \theta_a, & \text{if } h < \frac{\sqrt{5}}{5} \\ \theta_p, & \text{if } h > \frac{\sqrt{5}}{5} \end{cases}.$$

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